The Rho-Function

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Abstract

This mathematical essay discusses the function $\rho(x)$, which fullfills the equation $\rho'(x) = 2\rho(2x)$. It is infinitely smooth and can be used for splines. Its definition and some properties are presented.

1 Introduction

Positioning tasks for mechanical devices often demand to create a trajectory which allows for a smooth transition between two positions. If only the velocity of the (here one-dimensional) positioner is limited, the solution with minimal travelling time would obviously be to drive with maximum velocity until the target is reached.

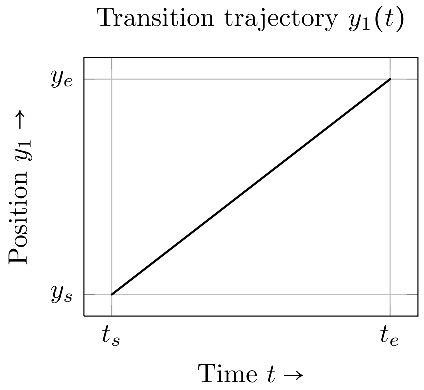

Figure 1: Time-optimal transition with limited velocity

Here, y_s denotes the start position, y_e the end position, t_s is the start time and t_e the time, on which the positioner has reached the target position. From this point on, these parameters will be set to $t_s = 0$, $t_e = 1$, $y_s = 0$ and $y_e = 1$ (without loss of generality). Normally, the time needed for the transition depends on the machine limits and can not be chosen, but since our machine is purely hypothetical, we pick its limits so that the required time for a transition from $y = 0$ to $y = 1$ is 1.

If not only the velocity (C^0 continuity), but also acceleration limits of the positioner have to be considered (C^1 continuity), a minimum time trajectory has to be put together from parabola and line segments: acceleration until the maximum velocity is reached, then staying on this velocity for a certain amount of time, followed by the deceleration phase which has to stop at the target position. We will assume that the maximum velocity will not be reached on the given track length, thus it is enough to consider the acceleration limit only.

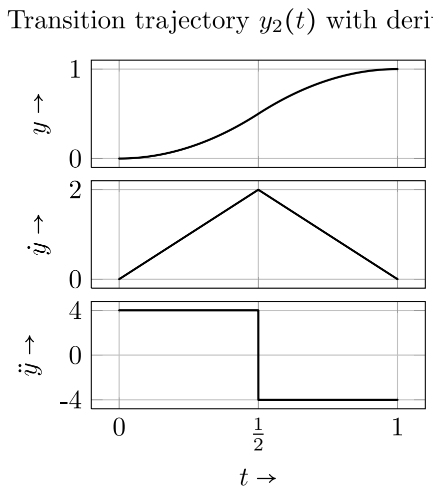

Figure 2: Time-optimal transition with limited acceleration

It is assumed that the machine stands still in the beginning and has to stop at the target position.

When high accuracy is demanded, it is reasonable to limit also the third derivative of the position (the jerk) in order to reduce oscillations. As before, lower derivative limits are no longer considered and all derivatives lower than the third have to be zero at $t = 0$ and at $t = 1$.

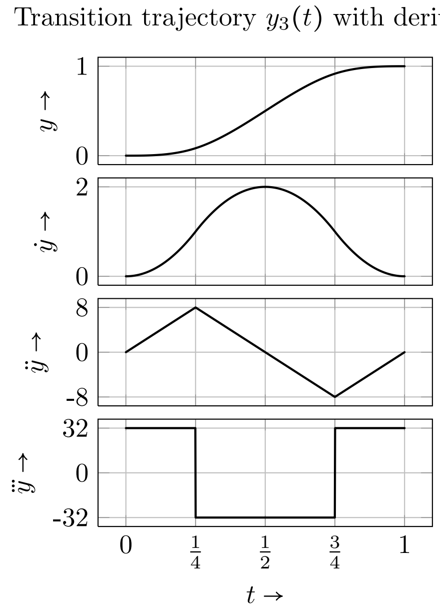

Figure 3: Time-optimal transition with limited jerk

Figure 3 shows the time-optimal transition when the jerk is limited (C^2 continuity). Note that $y_1(t)$ and $y_2(t)$ are already hard to distinguish for the eye.

For extremely high accuracy applications like nano-positioning machines, trajectories with steady jerk are desirable. This means that the fourth derivative of the position (referred to as jounce or snap) has to be limited (C^3 continuity). Figure 4 shows one possible trajectory with such a high degree of smoothness.

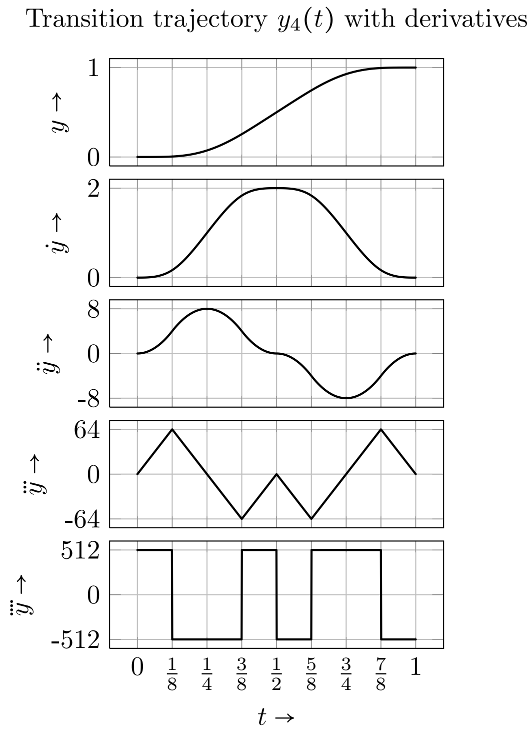

Figure 4: Transition with limited jounce

Note that this transition is no longer time-optimal; the central acceleration plateau is not necessary.

While we are at it: Maybe in future applications, e.g. for knitting two superstrings together, it will be required to limit the seventh derivative of the position.

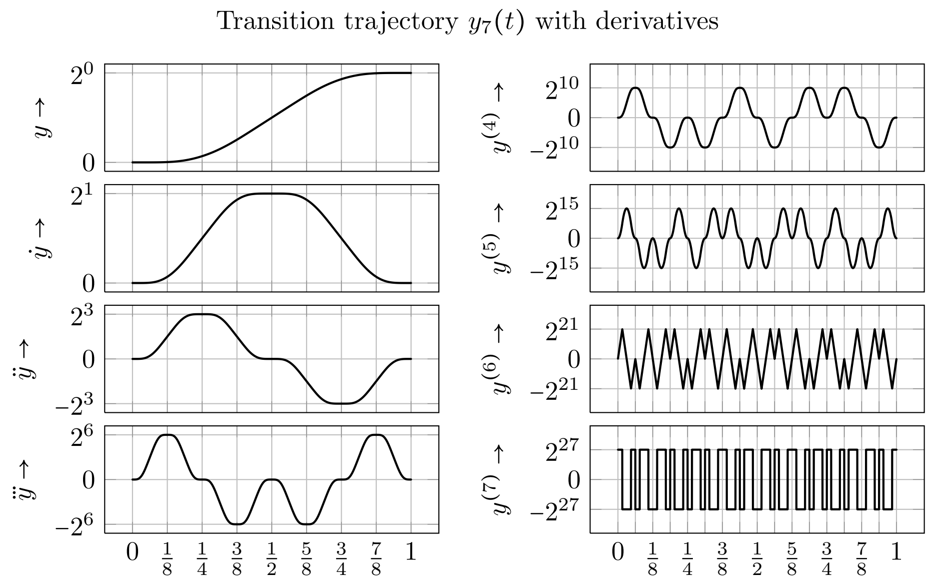

Figure 5: Transition with limited 7^{th} derivative

The function $y_i(t)$ appears to converge towards a limit function $\rho(t) = y_\infty(t)$ (ρ for ρllercoaster). Furthermore, the derivatives appear to converge towards the same function, just shrunk by a factor of $\frac{1}{2}$ in time and stretched by a factor of 2 in amplitude.

Purpose of this essay is to discuss this limit function.

2 Definition of the ρ-function

Derived from the observations of the previous chapter, the definition of the ρ function comes down to:

$$y_0(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le 2 \\ -y_0(x - 2^{\lfloor \log_2(x) \rfloor}), & \text{if } x > 2 \end{cases} \tag{1}$$

$$y_{i+1}(x) = \int_0^x 2y_i(2\tau)\mathrm{d}\tau \tag{2}$$

$$\rho(x) = y_\infty(x) \tag{3}$$

$2^{\lfloor \log_2(x) \rfloor}$ denotes the largest power of 2 that is smaller than or equal to x. To describe the definition of y_0 in words: The first part of the function from $x = 0$ to $x = 2$ is $\frac{1}{2}$; copy this part, flip it upside down and append it; copy the whole function from $x = 0$ to $x = 4$, flip it upside down and append it; copy the whole function from $x = 0$ to $x = 8$, flip it upside down and append it; ...

The generation of $y_{i+1}(x)$ from $y_i(x)$ smoothes the function with each integration by one degree, which makes $\rho(x)$ infinitely smooth.

While analyzing the properties of the function, a target will be to find an explicit or recursive formula to approximate $\rho(x)$ without the need of several integrations.

3 Properties of the ρ-function

The ρ-function shows several interesting properties. The first one is a direct consequence of its definition:

$$\rho'(x) = 2\rho(2x). \tag{4}$$

This can be considered the key property, most of the following properties can be derived from it.

3.1 Basic Properties

The ρ-function is

- defined only for positive arguments – In order to preserve the key property $\rho'(x) = 2\rho(2x)$, the definition can not be extended to cover negative arguments: If the first bump on the left side of $x = 0$ pointed upwards, then the first bump of its derivative on the left side of $x = 0$ would point downwards and vice versa, so the function would lose the property of being its own, scaled derivative.
- infinitely smooth – as already explained in the definition
- non-analytic – Approximating the ρ-function with a Taylor series in the neighborhood of $x = 0$ results in 0 for all x. A consequence from the definition is also that $\rho^{(a)}(\frac{n}{2^{(k-1)}}) = 0; a, n, k \in \mathbb{N}_0, a \ge k$, meaning that $\rho(x) = 0$ for $x = 2n$, $\rho'(x) = 0$ for $x = n$, $\rho''(x) = 0$ for $x = \frac{n}{2}$, $\rho'''(x) = 0$ for $x = \frac{n}{4}$ and so on; all further derivatives at these points are also 0. This entails, that $\rho(x)$ is also non-analytic at those points, and there is an infinite number of them on each arbitrarily small interval of the domain. An interesting question is, if the function is analytic at the rest of the domain.
- beautiful – how smoothly it starts to rise, then it goes into almost constant ascend, smoothly reducing the slope to zero again at the top of the bump. And there is perfect symmetry in all derivatives.

3.2 Symmetry

The recursive "copy-paste" generation law for y_0 (equation (1), second branch) can also be applied to any other $y_n(x)$, it is still valid after the scale and integration iteration step. It causes the ρ-function to have alternatingly even and odd symmetry with respect to powers of 2 for x. Note that this is only valid on an interval from $x = 0$ to twice the x-component of the regarded symmetry point/axis.

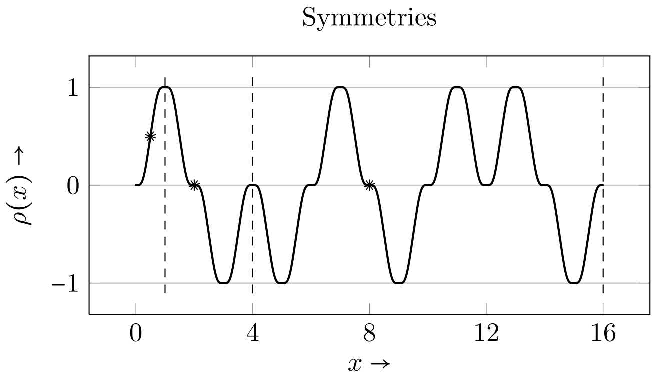

Figure 6: Symmetries of the ρ-function

Further symmetry can be found by shifting the derivatives. For instance, $\rho'(x)$ is symmetric with respect to the point $(\frac{1}{4}, 1)$. Hence, $\rho'(x) - 1$ is symmetric with respect to the point $(\frac{1}{4}, 0)$ and its antiderivative, $\rho(x) + x$ becomes symmetric with respect to the $x = \frac{1}{4}$ axis.

3.3 Function Value for certain Arguments

It is obvious that $\rho(0) = 0, \rho(\frac{1}{2}) = \frac{1}{2}, \rho(1) = 1$ and everything that can be concluded from symmetry properties for $\rho(\frac{n}{2}), n \in \mathbb{N}$. The first challenging point is $\rho(\frac{1}{4})$. Let us take a look at the sequence $y_i(\frac{1}{4})$, which can easily be calculated using a computer algebra system.

i	0	1	2	3	4	5	6	7	8	9	10
$y_i(\frac{1}{4})$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{7}{96}$	$\frac{9}{128}$	$\frac{107}{1536}$	$\frac{427}{6144}$	$\frac{569}{8192}$	$\frac{6827}{98304}$	$\frac{27307}{393216}$

A simple IQ-test-"Continue the series of numbers!"-analysis reveals the following pattern:

i	0	1	2	3	4	5	6	7	8	9	10
$y_i(\frac{1}{4})$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{7}{96}$	$\frac{9}{128}$	$\frac{107}{1536}$	$\frac{427}{6144}$	$\frac{569}{8192}$	$\frac{6827}{98304}$	$\frac{27307}{393216}$
$\Delta y_i = (y_i - y_{i+1})(\frac{1}{4})$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{96}$	$\frac{1}{384}$	$\frac{1}{1536}$	$\frac{1}{6144}$	$\frac{1}{24576}$	$\frac{1}{98304}$	$\frac{1}{393216}$	
$(\frac{\Delta y_i}{\Delta y_{i+1}})(\frac{1}{4})$	2	3	4	4	4	4	4	4	4		

Assuming that the last row of the table continues to be 4 for each $i > 1$, then $\rho(\frac{1}{4})$ can be calculated using the limit of the geometric series:

$$\rho\left(\frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{24}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{5}{72} \tag{5}$$

With the same (nonproven) approach, it can be calculated that $\rho(\frac{1}{8}) = \frac{1}{288}$ but it does no longer work for $\rho(\frac{1}{16})$. Using the symmetry properties (including derivative shifting as described in the symmetry section), further exact values can be obtained:

$$\rho\left(\frac{3}{8}\right) = \frac{3}{8} + \frac{1}{288} = \frac{109}{288} \tag{6}$$

$$\rho\left(\frac{5}{8}\right) = 1 - \rho\left(\frac{3}{8}\right) = \frac{179}{288} \tag{7}$$

$$\rho\left(\frac{3}{4}\right) = 1 - \rho\left(\frac{1}{4}\right) = \frac{67}{72} \tag{8}$$

$$\rho\left(\frac{7}{8}\right) = 1 - \rho\left(\frac{1}{8}\right) = \frac{287}{288} \tag{9}$$

3.4 Laplace Transformation

In Laplace domain the function fullfills the equation

$$sP(s) = 2P(2s). \tag{10}$$

We can calculate the following points:

$$P(0) = 0 \tag{11}$$

since the arithmetic mean of $\rho(t)$ is zero.

$$P(j\pi) = 0 \tag{12}$$

which can be seen in figure 7.

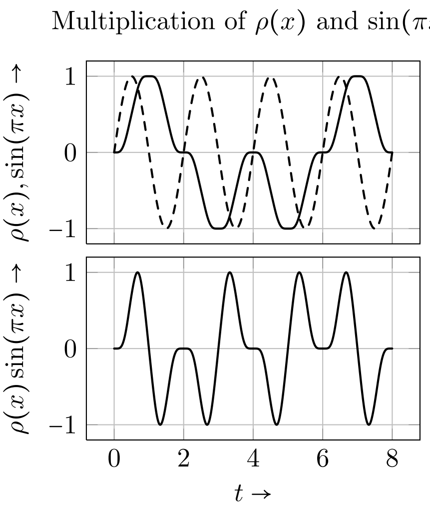

Figure 7: Construction of $P(j\pi)$

Each upward and downward peak of the ρ-function is symmetric about its center. Half of each peak is multiplied with the positive peak of a sine period, the other half with the negative peak. When integrated, the positive and negative parts compensate each other and the result is 0. Similarly:

$$P(jn\pi) = 0, \tag{13}$$

with $n \in \mathbb{N}$.

In this case, each upward and downward peak of the ρ-function is multiplied with an equal amount of full sine periods. Since there are as many upward peaks as there are downward peaks in the ρ-function, the integral from 0 to ∞ is 0.

From (10) we can conduct that

$$P(jn2^k\pi) = 0, \tag{14}$$

with $n \in \mathbb{N}, k \in \mathbb{Z}$. This is an interesting property, since $n2^k$ can approximate any positive real number with arbitrary accuracy.

3.5 Error Estimation

Since $\rho(x)$ can so far only be approximated by $y_i(x)$ for large i, it is worth knowing, how accurate this approximation is. Graphical analysis and incomplete induction suggest that the difference $|y_i(x) - \rho(x)|$ has its maxima at $x = \frac{1}{4} + \frac{n}{2}, n \in \mathbb{N}_0$ (independent of i and equal for all n). Since $\rho(\frac{1}{4})$ is known (or assumed), the maximum error can be computed by

$$\epsilon_i = \left|\rho\left(\frac{1}{4}\right) - y_i\left(\frac{1}{4}\right)\right| \tag{15}$$

$$= \frac{5}{72} - y_i\left(\frac{1}{4}\right) \tag{16}$$

A table of values for $y_i(\frac{1}{4})$ can be found in section 3.3, resulting in

i	0	1	2	3	4	5	6	7	8	9	10
ϵ_i	$\frac{31}{72}$	$\frac{13}{72}$	$\frac{1}{18}$	$\frac{1}{72}$	$\frac{1}{288}$	$\frac{1}{1152}$	$\frac{1}{4608}$	$\frac{1}{18432}$	$\frac{1}{73728}$	$\frac{1}{294912}$	$\frac{1}{1179648}$

4 Possible applications and further investigations

The ρ-function can be used to assemble infinitely smooth splines through arbitrary points with an arbitrary number of specified derivatives.

By convoluting with the first bump of the ρ-function, any function can be smoothened to C^∞-continuity.

However, some steps should be taken before the ρ-function can be used in good conscience.

A few conjectures were made, which have to be proven:

$$\rho\left(\frac{1}{4}\right) = \frac{5}{72} \tag{17}$$

and

$$\arg\max_{x \in \mathbb{R}^+} |\rho(x) - y_i(x)| = \frac{1}{2} - \frac{n}{4}, i, n \in \mathbb{N}. \tag{18}$$

In order to make this function more usable, a much simpler law of generation – preferably without integration – needs to be found. Finding further exact values, for instance $\rho(\frac{1}{16})$ could also be worth an investigation.

Another interesting task can be trying to expand the function for complex arguments.

Maybe it can also be shown, that

$$\int_0^\infty \rho(a\tau)\rho(b\tau)\mathrm{d}\tau = 0 \text{ for } a \neq b. \tag{19}$$

This could allow an operation similar to the Fourier-transformation, but with variations of $\rho(x)$ as basis functions instead of $\sin(x)$ and $\cos(x)$.